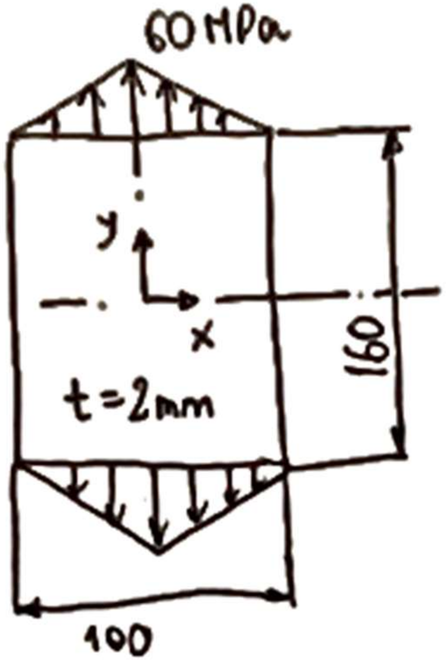
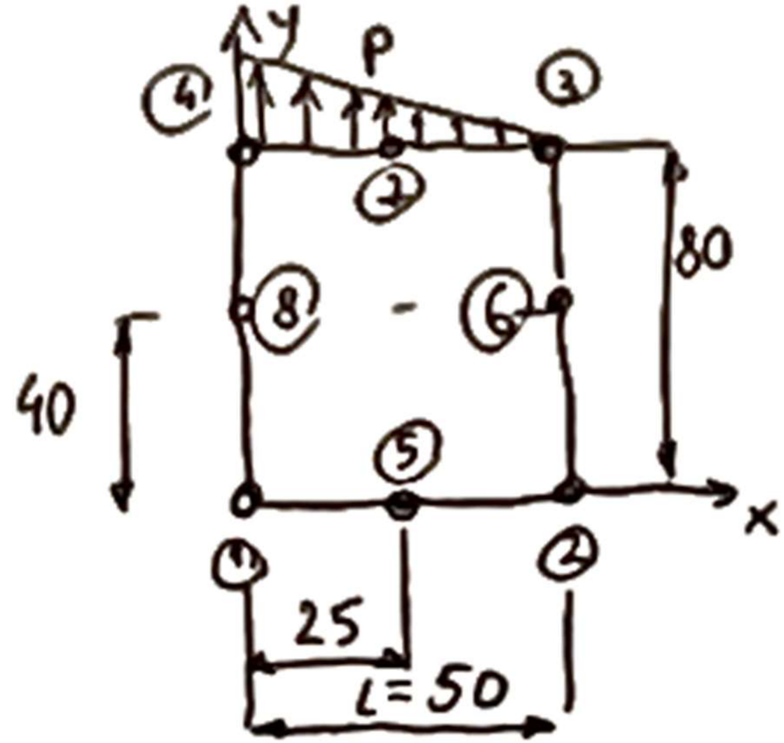
Example. FE model of 2D plate, QUAD-8node

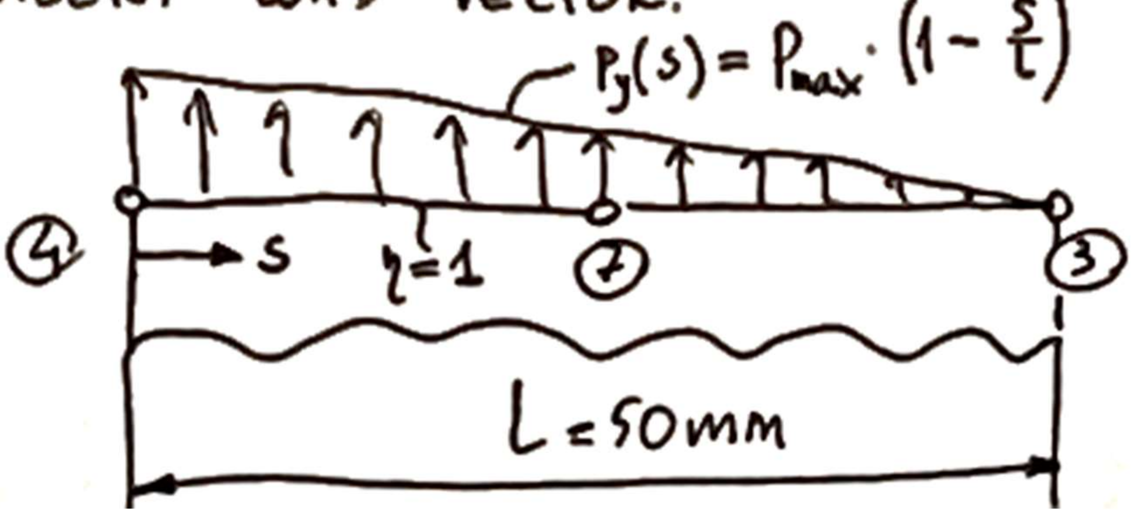
$E = 7.2 \cdot 10^4$ MPa
$\nu = 0.32$





$$\lfloor x_i \rfloor_1 = \lfloor 0, 50, 50, 0, 25, 50, 25, 0 \rfloor$$
$$\lfloor y_i \rfloor_1 = \lfloor 0, 0, 80, 80, 0, 40, 80, 40 \rfloor$$

EQUIVALENT LOAD VECTOR:



$x_3 = L \quad , \quad x_4 = 0 \quad , \quad x_7 = \frac{L}{2}$

$$\left.\begin{array}{l} s = 0 \Rightarrow \xi = -1 \\ s = L \Rightarrow \xi = 1 \end{array}\right\} \Rightarrow$$

$$s(\xi) = \frac{L}{2}(\xi + 1)$$

$$P_y(\xi) = P_{max}\left(1 - \frac{s(\xi)}{l}\right) = P_{max}\left(1 - \frac{1}{2}(\xi+1)\right) = \frac{P_{max}}{2}(1-\xi)$$

$$\lfloor F^P \rfloor_1 = t \cdot \int_{-1}^{1} \lfloor 0, p_y(\xi) \rfloor \cdot \underset{2\times 16}{[N]} \sqrt{\left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \cdot \underset{8\times 1}{\{x_i\}}\right)^2 + \left(\underset{1\times 8}{\frac{\partial \lfloor N(\xi,1) \rfloor}{\partial \xi}} \underset{8\times 1}{\{y_i\}}\right)^2}\, d\xi$$

$$N_1(\xi,\eta) = -\frac{1}{4}(1-\xi)(1-\eta)(1+\xi+\eta)$$

$$N_2(\xi,\eta) = -\frac{1}{4}(1+\xi)(1-\eta)(1-\xi+\eta)$$

$$N_3(\xi,\eta) = -\frac{1}{4}(1+\xi)(1+\eta)(1-\xi-\eta)$$

$$N_4(\xi,\eta) = -\frac{1}{4}(1-\xi)(1+\eta)(1+\xi-\eta)$$

$$N_5(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1-\eta)$$

$$N_6(\xi,\eta) = \frac{1}{2}(1+\xi)\left(1-\eta^2\right)$$

$$N_7(\xi,\eta) = \frac{1}{2}\left(1-\xi^2\right)(1+\eta)$$

$$N_8(\xi,\eta) = \frac{1}{2}(1-\xi)\left(1-\eta^2\right)$$

i	$\dfrac{\partial N_i}{\partial \xi}$	$\dfrac{\partial N_i}{\partial \eta}$
1	$\frac{1}{4}(1-\eta)(2\xi+\eta)$	$\frac{1}{4}(1-\xi)(\xi+2\eta)$
2	$\frac{1}{4}(1-\eta)(2\xi-\eta)$	$\frac{1}{4}(1+\xi)(2\eta-\xi)$
3	$\frac{1}{4}(1+\eta)(2\xi+\eta)$	$\frac{1}{4}(1+\xi)(\xi+2\eta)$
4	$\frac{1}{4}(1+\eta)(2\xi-\eta)$	$\frac{1}{4}(1-\xi)(2\eta-\xi)$
5	$-(1-\eta)\xi$	$-\frac{1}{2}(1-\xi^2)$
6	$\frac{1}{2}(1-\eta^2)$	$-(1+\xi)\cdot\eta$
7	$-(1+\eta)\xi$	$\frac{1}{2}(1-\xi^2)$
8	$-\frac{1}{2}(1-\eta^2)$	$-(1-\xi)\cdot\eta$

i	$\dfrac{\partial N_i(\xi,1)}{\partial \xi}$	$N_i(\xi,1)$
1	$\frac{1}{4}(1-1)\cdot(2\xi+1)=0$	0
2	$\frac{1}{4}(1-1)(2\xi-1)=0$	0
3	$\frac{1}{4}(1+1)(2\xi+1)=\frac{1}{2}(2\xi+1)$	$+\frac{1}{2}(1+\xi)\cdot\xi$
4	$\frac{1}{4}(1+1)(2\xi-1)=\frac{1}{2}(2\xi-1)$	$-\frac{1}{2}(1-\xi)\cdot\xi$
5	$-(1-1)\xi=0$	0
6	$\frac{1}{2}(1-1^2)=0$	0
7	$-(1+1)\xi=-2\xi$	$1-\xi^2$
8	$-\frac{1}{2}(1-1^2)=0$	0

$$\underset{1\times 8}{\frac{\partial\lfloor N(\xi,1)\rfloor}{\partial\xi}}\cdot\underset{8\times 1}{\{x_i\}_1} = \frac{\partial N_3(\xi,1)}{\partial\xi}\cdot x_3 + \frac{\partial N_4(\xi,1)}{\partial\xi}\cdot x_4 + \frac{\partial N_7(\xi,1)}{\partial\xi}\cdot x_7 =$$

$$= \frac{1}{2}(2\xi+1)\cdot L + \frac{1}{2}(2\xi-1)\cdot 0 - 2\xi\cdot\frac{L}{2} =$$

$$= \frac{L}{2}$$

$$\frac{\partial\lfloor N(\xi,1)\rfloor}{\partial\xi}\cdot\{y_i\}_1 = \frac{\partial N_3(\xi,1)}{\partial\xi}\cdot y_3 + \frac{\partial N_4(\xi,1)}{\partial\xi}\cdot y_4 + \frac{\partial N_7(\xi,1)}{\partial\xi}\cdot y_7 =$$

$$\left| \begin{array}{l} y_3 = y_4 = y_7 \\ = h \end{array} \right| \qquad = \left(\frac{1}{2}(2\xi+1) + \frac{1}{2}(2\xi-1) - 2\xi\right)\cdot h = 0$$

$$\Rightarrow \sqrt{\left(\frac{\partial\lfloor N(\xi,1)\rfloor}{\partial\xi}\cdot\{x_i\}\right)^2 + \left(\frac{\partial\lfloor N(\xi,1)\rfloor}{\partial\xi}\cdot\{y_i\}\right)^2} = \frac{L}{2}$$

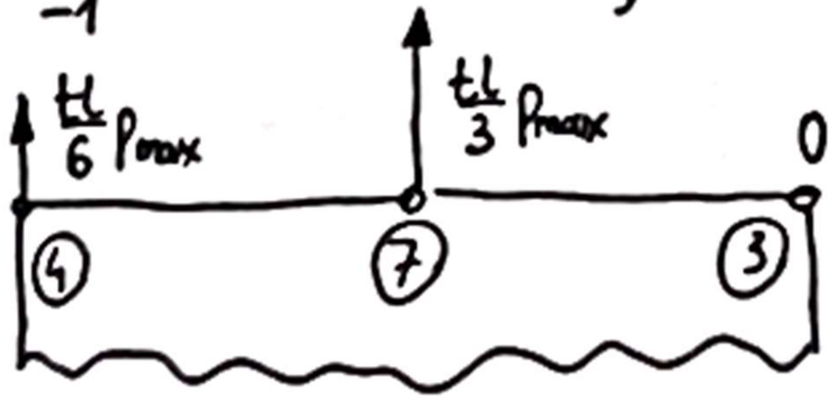
$$[F^p]_1 = \frac{tL}{2}\int_{-1}^{1} \lfloor 0, p_3(\xi) \rfloor \cdot [N(\xi,1)] \, d\xi$$

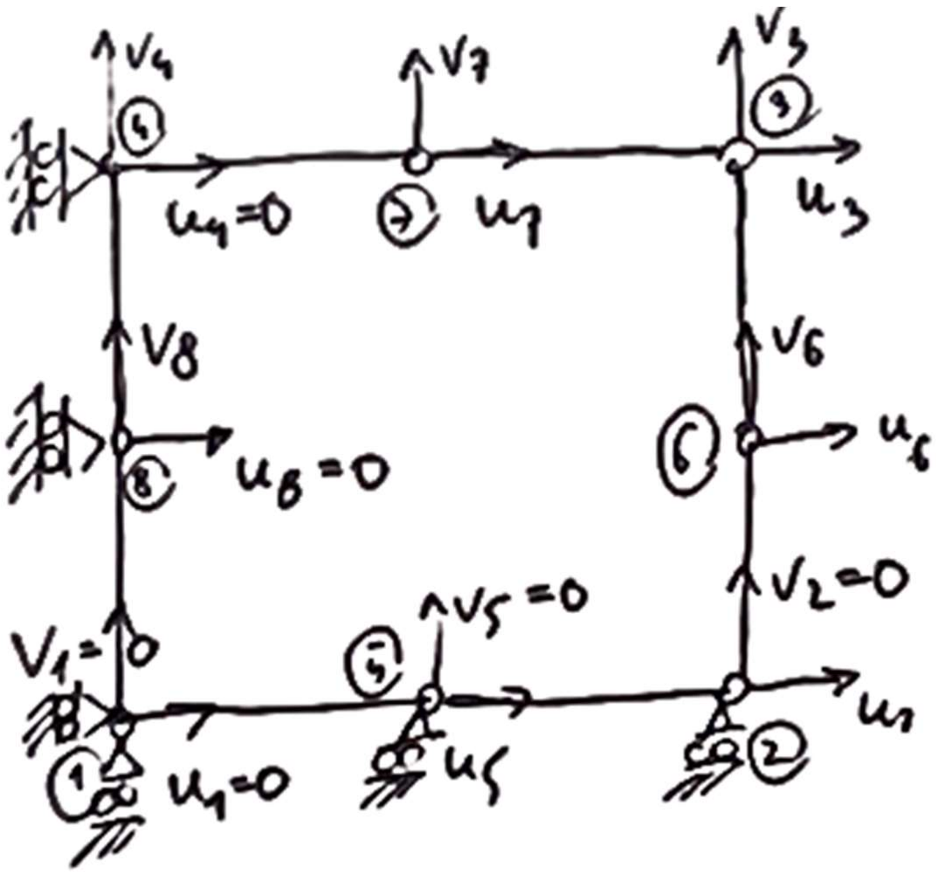
$$[N(\xi,1)] = \left[\begin{array}{c|c|c|c|c|c|c|c|c|c|c|c|c|c|c|c} 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) & 0 & 0 & 0 & 0 & 0 & N_7(\xi,1) & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & N_3(\xi,1) & 0 & N_4(\xi,1) & 0 & 0 & 0 & 0 & & N_7(\xi,1) & 0 & 0 \end{array}\right]$$

$$F_{61}^{p} = \frac{tL}{2}\int_{-1}^{1} \frac{p_{max}}{2}(1-\xi)\frac{1}{2}(1+\xi)\xi \, d\xi = 0$$

$$\bar{F}_{81}^{p} = \frac{tL}{2}\int_{-1}^{1} \frac{p_{max}}{2}(1-\xi)\left(-\frac{1}{2}(1-\xi)\xi\right) d\xi = \frac{tL}{6} p_{max}$$

$$F_{14\_1}^{p} = \frac{tL}{2}\int_{-1}^{1} \frac{p_{max}}{2}(1-\xi)\cdot(1-\xi^2)\, d\xi = \frac{tL}{3} p_{max}$$





$$\underset{1\times 16}{\lfloor q \rfloor} = \lfloor u_1, v_1, u_2, v_2, \ldots, u_8, v_8 \rfloor$$

BOUNDARY CONDITIONS : $N = 16 - 6 = 10$

$$\underset{1\times 10}{\lfloor q \rfloor} = \lfloor u_2, u_3, v_3, v_4, u_5, u_6, v_6, u_7, v_7, u_8 \rfloor$$

$$\underset{3\times 2}{[R]} = \begin{bmatrix} A_0\frac{\partial}{\partial\xi} + B_0\frac{\partial}{\partial\eta} & 0 \\ 0 & C_0\frac{\partial}{\partial\eta} + D_0\frac{\partial}{\partial\xi} \\ C_0\frac{\partial}{\partial\eta} + D_0\frac{\partial}{\partial\xi} & A_0\frac{\partial}{\partial\xi} + B_0\frac{\partial}{\partial\eta} \end{bmatrix} =$$

$$A_0 = \frac{1}{\det[J]}\frac{\partial y}{\partial\eta}, \quad B_0 = -\frac{1}{\det[J]}\cdot\frac{\partial y}{\partial\xi}$$

$$C_0 = \frac{1}{\det[J]}\frac{\partial x}{\partial\xi}, \quad D_0 = -\frac{1}{\det[J]}\cdot\frac{\partial x}{\partial\eta}$$

$$det[J] = det\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix} = \frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi}\frac{\partial x}{\partial \eta} =$$

$$= \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} - \frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} =$$

$$= \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{8\times 1}{\{x_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{8\times 1}{\{y_i\}_e}\right) - \left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{8\times 1}{\{y_i\}_e}\right)\left(\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{8\times 1}{\{x_i\}_e}\right)$$

$$\frac{\partial \xi}{\partial x} = \frac{1}{det[J]}\frac{\partial y}{\partial \eta} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \eta} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{8\times 1}{\{y_i\}_e}$$

$$\frac{\partial \eta}{\partial x} = -\frac{1}{det[J]}\frac{\partial y}{\partial \xi} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{y_i\}_e)}{\partial \xi} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{8\times 1}{\{y_i\}_e}$$

$$\frac{\partial \xi}{\partial y} = -\frac{1}{det[J]}\frac{\partial x}{\partial \eta} = -\frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \eta} = -\frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \eta}}\ \underset{8\times 1}{\{x_i\}_e}$$

$$\frac{\partial \eta}{\partial y} = \frac{1}{det[J]}\frac{\partial x}{\partial \xi} = \frac{1}{det[J]}\frac{\partial(\lfloor N(\xi,\eta)\rfloor\{x_i\}_e)}{\partial \xi} = \frac{1}{det[J]}\underset{1\times 8}{\frac{\partial\lfloor N(\xi,\eta)\rfloor}{\partial \xi}}\ \underset{8\times 1}{\{x_i\}_e}$$

$$[R] = \begin{bmatrix} A_0 \frac{\partial}{\partial \xi} & 0 \\ 0 & D_0 \frac{\partial}{\partial \xi} \\ D_0 \frac{\partial}{\partial \xi} & A_0 \frac{\partial}{\partial \xi} \end{bmatrix} + \begin{bmatrix} B_0 \frac{\partial}{\partial \eta} & 0 \\ 0 & C_0 \frac{\partial}{\partial \eta} \\ C_0 \frac{\partial}{\partial \eta} & B_0 \frac{\partial}{\partial \eta} \end{bmatrix} =$$

$$= \frac{\partial}{\partial \xi} \begin{bmatrix} A_0 & 0 \\ 0 & D_0 \\ D_0 & A_0 \end{bmatrix} + \frac{\partial}{\partial \eta} \begin{bmatrix} B_0 & 0 \\ 0 & C_0 \\ C_0 & B_0 \end{bmatrix}$$

$$\underset{3\times 16}{[B]} = \underset{3\times 2}{[R]} \cdot \underset{2\times 16}{[N]} = \begin{bmatrix} A_0 & 0 \\ 0 & D_0 \\ D_0 & A_0 \end{bmatrix} \underset{2\times 16}{\left[\frac{\partial N}{\partial \xi}\right]} + \begin{bmatrix} B_0 & 0 \\ 0 & C_0 \\ C_0 & B_0 \end{bmatrix} \cdot \underset{2\times 16}{\left[\frac{\partial N}{\partial \eta}\right]}$$

$$\underset{2\times 16}{\left[\frac{\partial N}{\partial \xi}\right]} = \begin{bmatrix} \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & 0 & \cdots & \frac{\partial N_8}{\partial \xi} & 0 \\ 0 & \frac{\partial N_1}{\partial \xi} & 0 & \frac{\partial N_2}{\partial \xi} & \cdots & 0 & \frac{\partial N_8}{\partial \xi} \end{bmatrix}$$

$$\underset{2\times 16}{\left[\frac{\partial N}{\partial \eta}\right]} = \begin{bmatrix} \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & 0 & \cdots & \frac{\partial N_8}{\partial \eta} & 0 \\ 0 & \frac{\partial N_1}{\partial \eta} & 0 & \frac{\partial N_2}{\partial \eta} & \cdots & 0 & \frac{\partial N_8}{\partial \eta} \end{bmatrix}$$

$$\underset{16\times 16}{[k]_e} = t_e \int_{-1}^{1}\int_{-1}^{1} \underset{16\times 3}{[B(\xi,\eta)]^T}\,\underset{3\times 3}{[D]}\,\underset{3\times 16}{[B(\xi,\eta)]}\,det[J]\,d\xi d\eta$$

Numerical integration (2 x 2 Gauss points)

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	157054.6148	48611.11111	88708.77897	259.9524658	78527.30739	20016.33987	52640.37433	-259.9524658	-218143.4442	-23915.62686	-42458.90275	-11437.9085	-114595.712	-11437.9085	-1733.016439	-21836.00713
2	48611.11111	101294.8109	-259.9524658	44529.8574	20016.33987	50647.40543	259.9524658	46635.47237	-21836.00713	-64589.52268	-11437.9085	-52753.0204	-11437.9085	-48541.79045	-23915.62686	-77223.21252
3	88708.77897	-259.9524658	157054.6148	-48611.11111	52640.37433	259.9524658	78527.30739	-20016.33987	-218143.4442	23915.62686	-1733.016439	21836.00713	-114595.712	11437.9085	-42458.90275	11437.9085
4	259.9524658	44529.8574	-48611.11111	101294.8109	-259.9524658	46635.47237	-20016.33987	50647.40543	21836.00713	-64589.52268	23915.62686	-77223.21252	11437.9085	-48541.79045	11437.9085	-52753.0204
5	78527.30739	20016.33987	52640.37433	-259.9524658	157054.6148	48611.11111	88708.77897	259.9524658	-114595.712	-11437.9085	-1733.016439	-21836.00713	-218143.4442	-23915.62686	-42458.90275	-11437.9085
6	20016.33987	50647.40543	259.9524658	46635.47237	48611.11111	101294.8109	-259.9524658	44529.8574	-11437.9085	-48541.79045	-23915.62686	-77223.21252	-21836.00713	-64589.52268	-11437.9085	-52753.0204
7	52640.37433	259.9524658	78527.30739	-20016.33987	88708.77897	-259.9524658	157054.6148	-48611.11111	-114595.712	11437.9085	-42458.90275	11437.9085	-218143.4442	23915.62686	-1733.016439	21836.00713
8	-259.9524658	46635.47237	-20016.33987	50647.40543	259.9524658	44529.8574	-48611.11111	101294.8109	11437.9085	-48541.79045	11437.9085	-52753.0204	21836.00713	-64589.52268	23915.62686	-77223.21252
9	-218143.4442	-21836.00713	-218143.4442	21836.00713	-114595.712	-11437.9085	-114595.712	11437.9085	458382.8481	-1.81899E-11	-1.81899E-11	-45751.63399	207095.4644	-9.09495E-12	5.45697E-12	45751.63399
10	-23915.62686	-64589.52268	23915.62686	-64589.52268	-11437.9085	-48541.79045	11437.9085	-48541.79045	-1.81899E-11	194167.1618	-45751.63399	7.27596E-12	-7.27596E-12	32095.46445	45751.63399	-3.63798E-12
11	-42458.90275	-11437.9085	-1733.016439	23915.62686	-1733.016439	-23915.62686	-42458.90275	11437.9085	-1.45519E-11	-45751.63399	169835.611	7.27596E-12	1.45519E-11	45751.63399	-81451.77263	5.45697E-12
12	-11437.9085	-52753.0204	21836.00713	-77223.21252	-21836.00713	-77223.21252	11437.9085	-52753.0204	-45751.63399	5.45697E-12	-7.27596E-12	211012.0816	45751.63399	-1.81899E-11	5.45697E-12	48940.38423
13	-114595.712	-11437.9085	-114595.712	11437.9085	-218143.4442	-21836.00713	-218143.4442	21836.00713	207095.4644	-9.09495E-12	0	45751.63399	458382.8481	-1.09139E-11	0	-45751.63399
14	-11437.9085	-48541.79045	11437.9085	-48541.79045	-23915.62686	-64589.52268	23915.62686	-64589.52268	-7.27596E-12	32095.46445	45751.63399	-1.09139E-11	-1.09139E-11	194167.1618	-45751.63399	0
15	-1733.016439	-23915.62686	-42458.90275	11437.9085	-42458.90275	-11437.9085	-1733.016439	23915.62686	2.00089E-11	45751.63399	-81451.77263	7.27596E-12	-2.91038E-11	-45751.63399	169835.611	0
16	-21836.00713	-77223.21252	11437.9085	-52753.0204	-11437.9085	-52753.0204	21836.00713	-77223.21252	45751.63399	-3.63798E-12	3.63798E-12	48940.38423	-45751.63399	0	1.09139E-11	211012.0816

[K] $_{NxN}$									
157054.6148	52640	259.9524658	-20016.33987	-2E+05	-1733	21836.00713	-1E+05	11437.9085	11438
52640.37433	157055	48611.11111	259.9524658	-1E+05	-1733	-21836.0071	-2E+05	-23915.6269	-11438
259.9524658	48611	101294.8109	44529.8574	-11438	-23916	-77223.2125	-21836	-64589.5227	-52753
-20016.33987	259.95	44529.8574	101294.8109	11438	11438	-52753.0204	21836	-64589.5227	-77223
-218143.4442	-1E+05	-11437.9085	11437.9085	458383	-2E-11	-45751.634	207095	-9.0949E-12	45752
-1733.016439	-1733	-23915.6269	11437.9085	-1E-11	169836	7.27596E-12	1E-11	45751.634	5E-12
21836.00713	-21836	-77223.2125	-52753.0204	-45752	-7E-12	211012.0816	45752	-1.819E-11	48940
-114595.712	-2E+05	-21836.0071	21836.00713	207095	0	45751.63399	458383	-1.0914E-11	-45752
11437.9085	-23916	-64589.5227	-64589.52268	-7E-12	45752	-1.0914E-11	-1E-11	194167.162	0
11437.9085	-11438	-52753.0204	-77223.21252	45752	4E-12	48940.38423	-45752	0	211012

$$\underset{10\times 1}{\{q\}} = \underset{10\times 10}{[K]^{-1}} \cdot \underset{10\times 1}{\{F\}}$$

F		q	
0		**-0.00523**	u2
0		**0.004468**	u3
0		**0.0181**	v3
1000	N	**0.046933**	v4
0		**-0.00353**	u5
0		**-0.01009**	u6
0		**0.013534**	v6
0		**0.001708**	u7
2000	N	**0.035171**	v7
0		**0.020224**	v8